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M. KRAWTCHOUK (1892–1942). PROFESSOR OF MATHEMATICAL STATISTICS

The Krawtchouk polynomials, a set of polynomials orthonormal with respect to the binomial distribution, are named after the Ukrainian mathematician Mikhailo Pylypovych Kravchuk. They were first exhibited in a Ukrainian-language publication (Kravchuk, 1929). An abbreviated version is the well-known paper: Krawtchouk (1929), taken up in Szegö (1939). Kravchuk worked as professor in charge of mathematics and variational statistics in the Kyiv (Kiev) Agricultural Institute 1921-1929, as well as elsewhere in this period. Elected to the Ukrainian Academy on 29 June 1929, from 1934-1938 he headed the Department of Mathematical Statistics of its Institute of Mathematics. He was arrested in 1938, and died near Magadan in Siberia.

We survey some probabilistic aspects of Kravchuk (1929, 1931) and Krawtchouk (1929), which, apart from the introduction of the celebrated orthonormal polynomials, contain expansions with respect to the binomial, estimates of the error in partial expansion, and a polynomial system orthonormal with respect to the hypergeometric.

1. Introduction

Kravchuk was a mathematician of broad interests and great productivity. The transliteration "Krawtchouk" of his name was the one he used when writing in French, and it is the transliteration with which Western readers are familiar which is why we have used it in the title. It seems more appropriate to cite his Ukrainian-language papers under the English transliteration "Kravchuk", which is the tranliteration therefore generally used throughout this article, except when speaking of the now-established : "Krawtchouk polynomials". His early work, under the influence of D. O. Gravè, was in algebra, but passed on to functions of a real variable and other areas of analysis, differential and integral equations, and geometry as well as mathematical statistics and probability. A theme running through much of his work is the method of moments. His statistical direction was initially driven by the work of Chebyshev on interpolation and the method

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of moments. In Kravchuk (1929) all the substantial papers of Chebyshev in this area (in turn derived from contact with the work of Irenée-Jules Bienaymé: Heyde and Seneta,1977), are cited. The motivation for statistical interests derived from Kravchuk's work in institutions such as the Kyiv Veterinary-Zootechnical Institute (in whose *Zapysky (Memoirs)* his first statistical paper was published in 1925) and the Agricultural Institute of Kyiv, in whose *Zapysky* the Krawtchouk polynomials make their first appearance.

His publications in mathematical statistics and probability, almost all in Ukrainian, are listed in Movchan (1972). They deal with the theory of correlation and regression, the bivariate normal, the method of moments in statistics (all influenced by the work of the then dominant English Biometric School, as is the early work of Slutsky) and orthogonal polynomial systems. His orthonormal polynomial system is excellently described, with the aftermath of its creation (including the work of Kravchuk's students O. S. (or: A. S.) Smohorshewsky, S. M. Kulik and O. K. Lebedintseva), in English in Prizva (1992). This description contains a quite detailed account of the major paper: Kravchuk (1931).

An overview of Kravchuk's work in English is contained in Parasyuk and Virchenko (1992). An account of his tragic last years is given in Syta (1993) with a brief sketch in English in Seneta (1993). These years exemplify Pascal's words: C'est un chose horrible de sentir s'écouler tout ce qu'on possède. (*Pensées*, 212).

An important and well-organized conference, which I attended, dedicated to the memory of Kravchuk on the 50th anniversary of his death was held in Kyiv in September,1992. There was an associated special issue of the *Ukrainian Mathematical Journal* (1992, 44, No. 7). The purpose of this note is to increase awareness of the man and his work among mathematical statisticians, as is appropriate at this conference. The discoverer of the Krawtchouk polynomials was, after all, a Professor of Mathematical Statistics.

There is a Scandinavian connection, since the Charlier polynomials (orthonormal with respect to the Poisson distribution) are a limiting case of the Krawtchouk ones, and these were first investigated in 1905 by the Swedish mathematician C. Charlier. There is also an Australian (specifically Sydney) connection within the work on biorthogonal polynomial expansions of bivariate distributions in the work of H. O. Lancaster and G. K. Eagleson.

2. The "Krawtchouk Polynomials" papers

The approach of Kravchuk (1929) has at its core the idea of interpolation by a polynomial function, using a weighted least squares approach, in a situation where observed points $\{y_i\}$ are equally spaced at $i = 0, 1, \ldots, u-1$. The weights $\{p_i\}$ in the Chebyshev situation are equal, and Kravchuk takes them to represent a uniform distribution on $0, 1, \ldots, u-1$. He considers the situation of arbitrary weights in his §1 and the Chebyshev situation in §2. In §3 he uses the general theory of §1 to produce the polynomials $\varphi_m(x, u; p, q), m = 0, 1, \ldots, u-1$ which we now call the Krawtchouk polynomials, after taking

$$p_i = P(i, u; p, q) = \binom{u-1}{i} p^i q^{u-1-i}, \qquad i = 0, 1, \ldots, u-1.$$

The first of these orthonormal polynomials are:

$$\varphi_0(x,u;p,q) = 1, \qquad \varphi_1(x,u;p,q) = \frac{p(u-1)-x}{(pq(u-1))^{1/2}}.$$

The paper is full of the ideas of what we now call the general linear model (e.g. residual sum of squares) and statistical notation, but the binomial weightings do not have any special significance in the context of the modern probabilistic distribution theory of the model. The summary makes clear that Kravchuk's focus is the fact that Chebyshev's polynomials are a discrete analogue of Legendre's, which could be obtained from them in the limit. He notes in his conclusion that, in the limit, one can get from his polynomials to the Hermite polynomials (orthonormal with respect to the standard normal density).

The *Comptes Rendus* paper (Krawtchouk,1929) is a résumé without proofs of Kravchuk (1929), and (to some degree) of a much more extensive outgrowth paper (Kravchuk,1931) presented to the Ukrainian Academy on 21 February, 1930, soon after his election. Kravchuk (1929) was received by its journal on 29 December,1928, before his election, and Krawtchouk (1929) was presented at the Séance du 23 septembre,1929, to the Académie des Sciences (by Emile Borel), after the election. Krawtchouk (1929) contains, as new items:

1. A new expression for the polynomial of degree $m, m = 0,1,\ldots,u-1$:

$$\varphi_m(x,u;p,q) = \left(\binom{u-1}{m}(pq)^{-m}\right)^{1/2} \sum_{i=0}^{m}(-1)^{m-i}\binom{u-x-1}{m-i}\binom{x}{i}p^{m-i}q^i,$$

   at $x = 0,1,\ldots,u-1$, which from its convolution structure invites a generating function approach, which is not taken up in these 3 papers, but occurs in Szegö (1939). The above expression is used to produce:
2. Charlier polynomials in the limit.
3. The evaluation in terms of Krawtchouk polynomials of generalized incomplete moments for $x \geq 1$:

$$\rho_m(x,u) = \sum_{i=0}^{x-1} P(i,u;p,q)\varphi_m(i,u;p,q), \qquad m \geq 0$$

   which in the case $m = 0$ is the distribution function of the binomial; and the incomplete factorial moments:

$$\sum_{i=0}^{x-1} P(i,u;p,q)\binom{p(u-1)-i}{k}, \qquad k \geq 0.$$

   The preoccupation with moments is characteristic of its time.
4. An expansion of one binomial probability in terms of another. This is given in somewhat general form in Krawtchouk (1929). The full account in Kravchuk (1931) shows that a simplified version is:

$$P(x,u_1;p_1,q_1) = P(x,u;p,q)\sum_{m=0}^{u-1} a_m\varphi_m(x,u;p,q), \qquad u_1 \leq u.$$

Krawtchouk (1929) notes the limiting forms: a Charlier expansion on the right (by putting $a = (u_1 - 1)p_1 = (u - 1)p$ and letting $u \to \infty$) of the left hand binomial probability; and an expansion in terms of Hermite polynomials. Clearly, his motivation, again, was to generalize these results.

3. The Hypergeometric distribution

After giving a full account of what we have described in our §2, Kravchuk (1) begins his §6 with the statement:

> Attempts have been made to base the theory of Pearson's distribution curves on the probabilistic scheme of sampling without replacement. In the box are $u-1$ white balls and $n-u$ black balls. Take out v balls, all at once. What is the probability $Q(x,u;n,v)$ that among them will be x white and $v-x$ black. As is known:

$$Q(x,u;n,v) = \frac{\binom{u-1}{x}\binom{n-u}{v-x}}{\binom{n-1}{v}}.$$

His general approach is to manipulate Q by introducing p and q so that it has the appearance of a function of binomial probabilities $P(i,w;p,q)$. He goes on to obtain an expansion for it in the Krawtchouk polynomials:

$$Q(x,u;n,v) = P(x,u;p,q)\sum_{m=0}^{u-1}\left(\frac{\binom{u-1}{m}}{\binom{n-1}{m}}\right)^{1/2}\varphi_m(v,n;p,q)\varphi_m(x,u;p,q)$$

and estimates the 'errors' in a partial expansion. For example, if $n \geq u+2$,and $v/(n-1)$ so that the means of the hypergeometric Q and the corresponding binomial P are matched up, he obtains:

$$|Q(x,u;n,v) - P(x,u;p,q)| \leq K\sqrt{((u-1)/v)}$$

which is of the form of the later well-developed work of Berry-Esséen type. The idea of an estimate of error is very much in the tradition of Chebyshev, whose work also contains polynomial expansions related to probability distributions.

In §8, he finally passes on to construct a system of polynomials *orthonormal with respect to the hypergeometric probabilities* $Q(x,u;n,v)$. It figures in the extended French summary (p. 47). The idea was taken up by his student Smohorshewsky, who presented a summary in *Comptes Rendus* (Smohorshewsky, 1935), but this was also not picked up in Western literature. The thinking of Kravchuk here was evidently probabilistic: inasmuch as the binomial probabilities $P(x,u;p,q)$ may be regarded as a limiting form of the hypergeometric probabilities $Q(x,u;n,v)$, the system so produced would be a generalization of the Krawtchouk polynomials.This polynomial system now enters within the later general system of Hahn polynomials (after W. Hahn). Certainly, in the hierarchy of polynomial systems defined by the hypergeometric functions, the Hahn polynomials are at a level higher than the Krawtchouk.

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